Value functions and nonsmooth analysis

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Value functions occur frequently in optimization theory and practice when a given problem depends on some parameters and one wishes to get information about solutions and their behaviors when the parameters are changed. Such functions fall outside the realm of classical analysis because they are nondifferentiable. They are of two types:

$$\text{performance functions}: \ p(w) := \inf_{x \in X} f(w,x),$$
$$\text{marginal functions}: \ m(w) := \sup_{x \in X} g(w,x).$$

Of course, $p = -m$ for $g := -f$. Here w is the parameter and x is the decision variable. However, since the objects replacing derivatives are of one-sided type, in general one cannot deduce results about performance functions from results about marginal functions and vice versa. These objects are of analytical nature (directional derivatives, subdifferentials, higher order generalized derivatives) or of geometrical nature (tangent cones, normal cones, higher order tangent sets). The interplay between these notions is a fascinating feature of nonsmooth analysis. We stress the importance of coherence in dealing with the various concepts (in terms of the processes of taking specific approximations which are substitutes for derivatives or tangent manifolds). We also show on various examples why taking advantage of a primal approach jointly with a dual approach is fruitful.

Among the examples one meets frequently are functions obtained through an infimal convolution (in particular distance functions and functions which regularize a given nonsmooth function). Such functions occur in particular in representations of solutions of Hamilton-Jacobi equations. Another large class of examples concerns value functions of parametrized mathematical programming problems. A huge literature has been devoted to this last topic under the name of sensitivity analysis. However, besides the convex case for which a well established duality theory has been developped, not much has been done about the interpretation of multipliers. It is our purpose to throw some light on such a question which has so many applications, especially in mathematical economics and in optimal control theory.