DECENTRALIZED CONVEX-TYPE EQUILIBRIUM IN NONCONVEX MODELS OF WELFARE ECONOMICS VIA NONLINEAR PRICES

BORIS S. MORDUKHOVICH [1]

Department of Mathematics, Wayne State University
Detroit, MI 48202, U.S.A.
boris@math.wayne.edu

Abstract. The paper is devoted to applications of modern tools of variational analysis to equilibrium models of welfare economics involving nonconvex economies with infinite-dimensional commodity spaces. The main results relate to *generalized/extended second welfare theorems* ensuring an equilibrium price support at Pareto optimal allocations. Based on advanced tools of generalized differentiation, we establish refined results of this type with the novel usage of *nonlinear prices* at the three types to optimal allocations: weak Pareto, Pareto, and strong Pareto. The usage of nonlinear (vs. standard linear) prices allow us to *decentralized* price equilibria in fully *nonconvex* models similarly to linear prices in the classical Arrow-Debreu convex model of welfare economics.

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1 Introduction

The classical Walrasian equilibrium model of welfare economics and its various generalizations have long been recognized as an important part of the economic theory and applications. It has been well understood that the concept of *Pareto efficiency/optimality* and its variants play a crucial role for the study of equilibria and making the best decisions for competitive economies; see, e.g., [2, 5, 6, 7, 9, 10, 14, 20, 22, 23, 33] and the references therein. After the pioneering work by Hicks, Lange, and Samuelson in the late 1930s and in the 1940s on the marginal rates of substitutions at Pareto optimal allocations, which lay at the foundations of welfare economics, the next crucial step was made in the beginning of 1950s by Arrow [6] and Debreu [10] for *convex* economies. Based on the classical *separation theorems* for convex sets, they and their followers developed a nice theory that, in particular, contains *necessary and sufficient* conditions for Pareto optimal allocations and shows that each of such allocations leads to a *decentralized equilibrium* in convex economies. The key result of this theory is now the classical *second fundamental theorem of welfare economics* stated that any Pareto optimal allocation can be *decentralized at price equilibria*, i.e., it can be sustained by a nonzero price vector at which each *consumer minimizes his/her expenditure* and each *firm maximizes its profit.* The full statement of this result is definitely *due to convexity*, which is crucial in the Arrow-Debreu model and its extensions. It is worth observing that the Arrow-Debreu general equilibrium theory of welfare economics has played an important role in the development of *convex analysis* as a mathematical discipline with its subsequent numerous applications.

On the other hand, the *relevance of convexity* assumptions is often doubtful for many important applications, which had been recognized even before developing the Arrow-Debreu model. It is

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well known, in particular, that convexity requirements do not hold in the presence of *increasing returns to scale* in the production sector. A common approach to the study of nonconvex models is based on utilizing local *convex tangent approximations* and then employing the classical separation theorems for convex cones. Constructively it has been done by using the Clarke tangent cone, which is *automatically convex*. In this way, marginal prices are formalized via Clarke's normal cone that, however, may be *too large* for satisfactory results in nonconvex models; the reader can find many examples, discussions, and references in the paper by Khan [20]. The latter paper contains much more adequate extensions of the second welfare theorem to nonconvex economies with *finite-dimensional* commodity spaces, where marginal prices are formalized via the *nonconvex* normal cone introduced by Mordukhovich [24].

In our previous work on the extended second welfare theorem(s) for nonconvex models [26, 27, 22] we developed an approach based on the *extremal principle* of variational analysis, which can be treated as a *variational counterpart* of the classical separation in the case of nonconvex sets and which plays essentially the same role in nonconvex variational analysis as separation theorems do in the convex framework; see the survey paper by Mordukhovich [27] and also the book [29] for more information. On the other hand, the extremal principle provides necessary conditions for extremal points of nonconvex sets that particularly cover the case of Pareto-type optimal allocations. We refer the reader to other recent publications [8, 13, 14, 16, 17, 28, 34] that explore some related nonconvex separation properties in applications to various nonconvex models of welfare economics in both finite-dimensional and infinite-dimensional frameworks.

The machinery of the extremal principle allows us to derive extended versions of the second welfare theorem for nonconvex economies in both *approximate/fuzzy* and *exact/limiting* forms under mild net demand qualification conditions needed in conventional cases of *Pareto* and *weak Pareto* optimal allocations. In this way we establish efficient conditions ensuring the marginal price *positivity* when commodity spaces are *ordered*. The results obtained bring new information even in the case of *convex economies*, since we *do not impose* either the classical *interiority* condition or the widely implemented *properness* condition by Mas-Colell [23]. Moreover, in contrast to the vast majority of publications on convex economies with ordered commodity spaces, our approach *does not require a lattice structure* of the commodity space in question.

In this paper, we further develop the approach based on the extremal principle to nonconvex economies with infinite-dimensional commodity spaces. Among other developments, the *main emphasis* now goes to the following issues, for which the results obtained seem to be new and useful even for model with finite-dimensional commodities:

—introducing *nonlinear prices* to describe an approximate *decentralized* equilibrium of the convex type in *nonconvex* economies for all the three basic versions of Pareto optimality mentioned above, under the appropriate geometric assumptions on the commodity space in question;

—clarifying a surprisingly remarkable role of *strong Pareto* optimality in the context of extended second welfare theorems for economies with ordered commodity spaces, which *do not require* any net demand (interiority type) qualification conditions even in the classical convex settings.

2 Basic Model and Concepts of Welfare Economics

Let E be a normed *commodity space* of the economy $\mathcal{E}$ that involves $n \in I\!N := \{1, 2, \ldots\}$ consumers with *consumption sets* $C_i \subset E$, $i = 1, \ldots, n$, and $m \in I\!N$ firms with *production sets* $S_j \subset E$, $j = 1, \ldots, m$. Each consumer has a *preference set* $P_i(x)$ that consists of elements in C_i preferred to x_i by this consumer at the consumption plan/bundle $x = (x_1, \ldots, x_n) \in C_1 \times \cdots \times C_n$. This is a useful generalization (with valuable economic interpretations) of standard ordering relations given, in particular, by utility functions as in the classical models of welfare economics. We have by definition that $x_i \notin P_i(x)$ for all $i = 1, \ldots, n$ and always assume that $P_i(x) \neq \emptyset$ for some $i \in \{1, \ldots, n\}$. For convenience we put $\mathrm{cl}\, P_i(x) := \{x_i\}$ if $P_i(x) = \emptyset$.

Now define *feasible allocations* of the economy $\mathcal{E}$ imposing *market constraints* formalized via a given subset $W \subset E$ of the commodity space; we label W as the *net demand constraint set* in $\mathcal{E}$.

Definition 2.1 (feasible allocations). *Let $x = (x_i) := (x_1, \ldots, x_n)$, and let $y = (y_j) := (y_1, \ldots, y_m)$. The pair $(x, y) \in \prod_{i=1}^{n} C_i \times \prod_{j=1}^{m} S_j$ is called a* FEASIBLE ALLOCATION *of $\mathcal{E}$ if*

$$w := \sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j \in W. \tag{2.1}$$

Introducing the net constraint set as in [26] allows us to unify some conventional situations in economic models and to give a useful economic insight in the general framework. Indeed, in the classical case the set W consists of one element $\{\omega\}$, where ω is an *aggregate endowment* of scarce resources. Then constraint (2.1) reduces to the *markets clear* condition. Another conventional framework appears in (2.1) when the commodity space E is ordered by a closed positive cone E_+ and we put $W := \omega - E_+$, which corresponds to the *implicit free disposal* of commodities. Generally constraint (2.1) describes a natural situation that may particularly happen when the initial aggregate endowment is not exactly known due to, e.g., *incomplete information.* In the latter general case the set W reflects some *uncertainty* in the economic model under consideration.

In this paper we consider the following three Pareto-type notions of optimality for feasible allocations in the economic model $\mathcal{E}$: *weak Pareto* optimality, *Pareto* optimality, and *strong Pareto* optimality. The first two notions have been well recognized and developed in the economic literature; they go back to the classical Pareto and weak Pareto concepts in vector/multiobjective optimization defined via utility functions. To the best of our knowledge, the notion of strong Pareto optimality in models of welfare economics was first introduced and studied by Khan [19]. Its special role in the framework of second welfare theorems was observed by Mordukhovich [26, 27]; this will be further developed in the present paper.

Definition 2.2 (Pareto-type optimal allocations). *Let $(\bar{x}, \bar{y})$ be a feasible allocation of the economy $\mathcal{E}$ with the property*

$$\bar{x}_i \in \mathrm{cl}\, P_i(\bar{x}) \ \text{ for all } \ i = 1, \ldots, n.$$

It is said that:

(i) *$(\bar{x}, \bar{y})$ is a local* WEAK PARETO OPTIMAL ALLOCATION *of $\mathcal{E}$ if there is a neighborhood O of $(\bar{x}, \bar{y})$ such that for every feasible allocation $(x, y) \in O$ one has $x_i \notin P_i(\bar{x})$ for some $i \in \{1, \ldots, n\}$.*

(ii) *$(\bar{x}, \bar{y})$ is a local* PARETO OPTIMAL ALLOCATION *of $\mathcal{E}$ if there is a neighborhood O of $(\bar{x}, \bar{y})$ such that for every feasible allocation $(x, y) \in O$ either $x_i \notin \mathrm{cl}\, P_i(\bar{x})$ for some $i \in \{1, \ldots, n\}$ or $x_i \notin P_i(\bar{x})$ for all $i = 1, \ldots, n$.*

(iii) *$(\bar{x}, \bar{y})$ is a local* STRONG PARETO OPTIMAL ALLOCATION *of $\mathcal{E}$ is there is a neighborhood O of $(\bar{x}, \bar{y})$ such that for every feasible allocation $(x, y) \in O$ with $(x, y) \neq (\bar{x}, \bar{y})$ one has $x_i \notin \mathrm{cl}\, P_i(\bar{x})$ for some $i \in \{1, \ldots, n\}$.*

To deal with weak Pareto and Pareto optimal allocations, we need some *qualification conditions.* The following ones were formulated and studied in [26, 27, 22]; they are in the line of "asymptotically including conditions" by Jofré [16] and Jofré and Rivera [17] (for $W = \{\omega\}$) and their early versions in Cornet [9] and Khan [19, 20]; cf. also the references and discussions therein.

Definition 2.3 (net demand qualification conditions). *Let $(\bar{x}, \bar{y})$ be a feasible allocation of the economy $\mathcal{E}$, and let*

$$\bar{w} := \sum_{i=1}^{n} \bar{x}_i - \sum_{j=1}^{m} \bar{y}_j. \tag{2.2}$$

Given $\varepsilon > 0$, we consider the set

$$\Delta_\varepsilon := \sum_{i=1}^{n} \operatorname{cl} P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon I\!B) - \sum_{j=1}^{m} \operatorname{cl} S_j \cap (\bar{y}_j + \varepsilon I\!B) - \operatorname{cl} W \cap (\bar{w} + \varepsilon I\!B)$$

and say that:

(i) *The* NET DEMAND QUALIFICATION (NDQ) CONDITION *holds at $(\bar{x}, \bar{y})$ if there are $\varepsilon > 0$, a sequence $\{e_k\} \subset X$ with $e_k \to 0$ as $k \to \infty$, and a consumer index $i_0 \in \{1, \ldots, n\}$ such that*

$$\Delta_\varepsilon + e_k \subset P_{i_0}(\bar{x}) + \sum_{i \neq i_0} \operatorname{cl} P_i(\bar{x}) - \sum_{j=1}^{m} S_j - W$$

for all $k \in I\!N$ sufficiently large.

(ii) *The* NET DEMAND WEAK QUALIFICATION (NDWQ) CONDITION *holds at $(\bar{x}, \bar{y})$ if there are $\varepsilon > 0$ and a sequence $e_k \to 0$ as $k \to \infty$ such that*

$$\Delta_\varepsilon + e_k \subset \sum_{i=1}^{n} P_i(\bar{x}) - \sum_{j=1}^{m} S_j - W$$

for all $k \in I\!N$ sufficiently large.

Observe that the qualification conditions from Definition 2.3 hold of *either one* among preference or production sets is *epi-Lipschitzian* around the corresponding point in the sense of Rockafellar; see Section 3. It is well known that for *convex* sets Ω the epi-Lipschitzian property of Ω is equivalent to $\operatorname{int} \Omega \neq \emptyset$. Thus the above qualification conditions may be viewed as far-going extensions of the classical nonempty interiority condition well-developed for convex models of welfare economics. We refer the reader to [22, 29] for more general sufficient conditions ensuring the fulfillment of both NDQ and NDWQ properties. It follows from those results that *no* assumptions on either preference or production sets are needed to ensure the above qualification conditions provided that the net demand constraint set W is epi-Lipschitzian around the point $\bar{w}$ defined in (2.2). The latter covers particularly the case of "free-disposal Pareto optimum" studied by Cornet [9] for nonconvex economies with finite-dimensional commodity spaces.

3 Tools of Variational Analysis

This section contains some constructions and results from variational analysis and generalized differentiation that play the crucial role in the subsequent extensions of the second welfare theorem with nonlinear price descriptions of decentralized equilibria. We start with the basic generalized differential constructions referring the reader to the books of Rockafellar and Wets [32] and Mordukhovich [29] for more details in finite dimensions [32] and in both finite-dimensional and infinite-dimensional spaces [29]. Recall that, given a set-valued mapping $F\colon X \rightrightarrows X^*$ from a Banach space X to its dual space X^* endowed with the weak* topology w^*, the notation

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \Big\{ x^* \in X^* \Big| \;\; \exists \text{ sequences } \; x_k \to \bar{x} \; \text{ and } x_k^* \xrightarrow{w^*} x^* \\ \text{such that } \; x_k^* \in F(x_k) \; \text{ for all } \; k \in I\!N \Big\}$$

stands for the *sequential Kuratowski-Painlevé upper/outer limit* of F as $x \to \bar{x}$.

Definition 3.1 (generalized normals). *Let $\Omega \subset X$ be a nonempty subset of a Banach space, and let $\varepsilon \ge 0$.*

(i) *Given $x \in \Omega$, define the* SET OF ε-NORMALS *to Ω at x by*

$$\widehat{N}_\varepsilon(x;\Omega) := \Big\{x^* \in X^* \Big| \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon\Big\}, \tag{3.1}$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. When $\varepsilon = 0$, the set (3.1) *is a convex cone called the* PRENORMAL CONE *or the* FRÉCHET NORMAL CONE *to Ω at x and denoted by $\widehat{N}(x;\Omega)$. If $x \notin \Omega$, put $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \ge 0$.*

(ii) *The conic set*

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega) \tag{3.2}$$

is called the (*basic, limiting*) NORMAL CONE *to Ω at $\bar{x} \in \Omega$.*

In the finite-dimensional case $X = I\!R^n$, the basic (often nonconvex) normal cone (3.2) reduces to the one introduced by Mordukhovich [24] as

$$N(\bar{x};\Omega) = \mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} [\mathrm{cone}(x - \Pi(x;\Omega))], \tag{3.3}$$

where "cone" stands for the conic hull of a set, and where $\Pi(x;\Omega)$ is the multivalued Euclidean projector of x on the closure of Ω. The set of ε-normals (3.1) and the extension (3.2) of the basic normal cone to Banach spaces first appeared in Kruger and Mordukhovich [21].

If $\Omega \subset X$ is *convex*, then for all $\varepsilon \ge 0$ one has

$$\begin{cases} \widehat{N}_\varepsilon(\bar{x};\Omega) = \widehat{N}(\bar{x};\Omega) + \varepsilon I\!B^* = \{x^* \in X^* |\ \langle x^*, x - \bar{x}\rangle \le \varepsilon\|x - \bar{x}\| \ \text{ whenever } \ x \in \Omega\}, \\ N(\bar{x};\Omega) = \{x^* \in X^* |\ \langle x^*, x - \bar{x}\rangle \le 0 \ \text{ whenever } \ x \in \Omega\}, \end{cases} \tag{3.4}$$

where $I\!B^* \subset X^*$ (and likewise $I\!B \subset X$) stands for the closed unit ball in the space in question. This implies that both prenormal and normal cones from Definition 3.1 reduce to the normal cone of convex analysis for convex sets in Banach spaces.

Despite of (in fact due to) its nonconvexity, the basic normal cone (3.1) and associated subdifferential and coderivative constructions of extended-real-valued functions and set-valued mappings possess many useful properties in general Banach space settings; see [29]. However, the most reliable framework for the theory and applications of (3.1) is the realm of *Asplund spaces*, which can be equivalently defined as Banach spaces whose separable subspaces have separable duals. This class particularly includes all *reflexive* spaces; see, e.g., Phelps' book [31] for more details and references. It has been proved in [30], based on variational arguments, that

$$N(\bar{x};\Omega) = \mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} \widehat{N}(x;\Omega) \tag{3.5}$$

when X is Asplund and when Ω is locally closed around $\bar{x}$, i.e., one can equivalently put $\varepsilon = 0$ in (3.2). Note also that the weak* convex closure $\mathrm{cl}^*\mathrm{co}\, N(\bar{x};\Omega)$ of the basic normal cone agrees with the Clarke normal cone in this setting; see [29, 30] for more details.

Recall that a Banach space X admits a *Fréchet smooth renorm* if there is an equivalent norm on X that is Fréchet differentiable at any nonzero point. In particular, every reflexive space admits a Fréchet smooth renorm. We also consider Banach spaces admitting an *$\mathcal{S}$-smooth bump function* with respect to a given class $\mathcal{S}$, i.e., a function $b: X \to I\!R$ such that $b(\cdot) \in \mathcal{S}$, $b(x_0) \neq 0$ for some $x_0 \in X$, and $b(x) = 0$ whenever x lies outside a ball in X. In what follows we deal with the three classes of $\mathcal{S}$-smooth functions on X: Fréchet smooth ($\mathcal{S} = \mathcal{F}$), Lipschitzian and Fréchet smooth ($\mathcal{S} = \mathcal{LF}$), and Lipschitzian and continuously differentiable ($\mathcal{S} = \mathcal{LC}^1$). It is well known that the class of spaces admitting a $\mathcal{LC}^1$-smooth bump function strictly includes the class of spaces with a Fréchet smooth renorm. Observe that all the spaces listed above are Asplund.

Theorem 3.2 (smooth variational descriptions of Fréchet normals). *Let Ω be a nonempty subset of a Banach space X, and let $\bar{x} \in \Omega$. The following hold:*

(i) *Given $x^* \in X^*$, we assume that there is a function $s: U \to I\!R$ defined on a neighborhood of $\bar{x}$ and Fréchet differentiable at $\bar{x}$ such that $\nabla s(\bar{x}) = x^*$ and $s(x)$ achieves a local maximum relative to Ω at $\bar{x}$. Then $x^* \in \widehat{N}(\bar{x};\Omega)$. Conversely, for every $x^* \in \widehat{N}(\bar{x};\Omega)$ there is a function $s: X \to I\!R$ such that $s(x) \le s(\bar{x}) = 0$ whenever $x \in \Omega$ and that $s(\cdot)$ is Fréchet differentiable at $\bar{x}$ with $\nabla s(\bar{x}) = x^*$.*

(ii) *Assume that X admits a Fréchet smooth renorm. Then for every $x^* \in \widehat{N}(\bar{x};\Omega)$ there is a concave Fréchet smooth function $s: X \to I\!R$ that achieves its global maximum relative to Ω uniquely at $\bar{x}$ and such that $\nabla s(\bar{x}) = x^*$.*

(iii) *Assume that X admits an $\mathcal{S}$-smooth bump function, where $\mathcal{S}$ stands for one of the classes $\mathcal{F}$, $\mathcal{LF}$, or $\mathcal{LC}^1$. Then for every $x^* \in \widehat{N}(\bar{x};\Omega)$ there is an $\mathcal{S}$-smooth function $s: X \to I\!R$ satisfying the conclusions in* (ii).

Proof. Under the assumptions in (i) one has

$$s(x) = s(\bar{x}) + \langle x^*, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \le s(\bar{x})$$

for all $x \in \Omega$ near $\bar{x}$. Hence we get

$$\langle x^*, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \le 0 \quad \text{for such} \quad x,$$

which implies that $x^* \in \widehat{N}(\bar{x};\Omega)$ due to Definition 3.1(i) with $\varepsilon = 0$.

To justify the converse statement in (i), pick and $x^* \in \widehat{N}(\bar{x};\Omega)$ and define the function

$$s(x) := \begin{cases} \min\{0, \langle x^*, x - \bar{x} \rangle\} & \text{if } x \in \Omega, \\ \langle x^*, x - \bar{x} \rangle & \text{otherwise.} \end{cases}$$

It follows from the construction of Fréchet normals in (3.1) as $\varepsilon = 0$ that this function is Fréchet differentiable at $\bar{x}$ with $\nabla s(\bar{x}) = x^*$. Moreover, one clearly has by definition of $s(\cdot)$ that $s(\bar{x}) = 0$ and that $s(x) \le 0$ whenever $x \in \Omega$, which completes the proof of (i).

Assertions (ii) and (iii) follow from the subdifferential results of [11, Theorem 4.6] whose proof is much more involved; see also the proof of [29, Theorem 1.30] for some simplifications. $\triangle$

The basic geometric result of variational analysis is the *extremal principle*, which is crucial in our applications to the extended second welfare theorems. Given a common point $\bar{x}$ of two sets $\Omega_1, \Omega_2 \subset X$ in a Banach space, we say that $\bar{x}$ is a *locally extremal point* of the set system $\{\Omega_1, \Omega_2\}$ if there exists a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ there is $a \in \varepsilon I\!B$ with $(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset$. The next theorem gives a version of the extremal principle used in what follows; see Mordukhovich [27, 29] for the proof, more discussions, and various applications.

Theorem 3.3 (extremal principle). *Let $\bar{x} \in \Omega_1 \cap \Omega_2$ be a locally extremal point of the two closed subsets of the Asplund space X. Then for every $\varepsilon > 0$ there are $x_i \in \Omega_i \cap (\bar{x} + \varepsilon I\!B)$, $i = 1, 2$, and $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$x^* \in (\widehat{N}(x_1; \Omega_1) + \varepsilon I\!B^*) \cap (- \widehat{N}(x_2; \Omega_2) + \varepsilon I\!B^*).$$

For implementing limiting procedures in the extremal principle (as $\varepsilon \downarrow 0$) and related results of variational analysis, we need some *normal compactness* properties of sets and set-valued mappings. Recall that a set $\Omega \subset X$ is *sequentially normally compact*, or *SNC*, at $\bar{x} \in \Omega$ if for any sequences $(x_k, x_k^*) \in X \times X^*$ satisfying

$$x_k^* \in \widehat{N}(x_k; \Omega), \quad x_k \to \bar{x}, \text{ and } x_k^* \xrightarrow{w^*} \bar{x}$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$; cf. [30, 29] and the references therein. This property, which is obviously automatic in finite dimensions, can be viewed as an extension of the finite codimension property to nonconvex sets. It is closely related to, being generally weaker than, the *compactly epi-Lipschitzian* (CEL) property of arbitrary sets $\Omega \subset X$ around $\bar{x} \in \Omega$ in the following sense of Borwein-Strojwas: there are a compact set $C \subset X$, a neighborhood O of the origin in X, and a number $\gamma > 0$ such that

$$\Omega \cap U + tO \subset \Omega + tC \text{ for all } t \in (0, \gamma). \tag{3.6}$$

The case of a singleton C in (3.6) corresponds to the *epi-Lipschitzian* property of Ω around $\bar{x}$ in the sense of Rockafellar. A thorough study of the CEL property can be found in Ioffe [15], while comprehensive relationships between the SNC and CEL properties have been recently established by Fabian and Mordukhovich [12].

4 Extended Second Welfare Theorems

In this section we develop results on necessary conditions for Pareto and weak Pareto optimal allocations of the nonconvex economy $\mathcal{E}$ with an *Asplund* commodity space E *without* imposing *any ordering* structure on commodities. First we derive refined versions of the extended second welfare theorem in *approximate* forms involving ε-equilibrium prices that support (local) Pareto and weak Pareto *suboptimal* allocations. The obtained ε-equilibrium prices admit two types of *equivalent* descriptions: either as *marginal linear prices* formalized via Fréchet-like normals, or as *nonlinear prices* supporting a *decentralized* (i.e., convex-type) equilibrium in fully nonconvex settings; see below. Note that the idea of using nonlinear prices in somewhat different frameworks have been recently suggested and developed in the economic literature by Aliprantis et al. [3, 4, 5]. For the reader's information, we mention that in [5] an alternative notion of nonlinear prices is proposed for convex economies; in [4] it is shown that such nonlinear prices are different from linear ones even for convex economies with three-dimensional commodity spaces; and the last paper [3] characterizes the major differences between linear and nonlinear prices introduced and studied in this trilogy.

Theorem 4.1 (approximate marginal and decentralized forms of the extended second welfare theorem for Pareto and weak Pareto optimal allocations). *Let the pair $(\bar{x}, \bar{y})$ be a local Pareto (resp. weak Pareto) optimal allocation of the economy $\mathcal{E}$ with an Asplund commodity space E. Assume that the net demand qualification condition (resp. net demand weak qualification condition) is satisfied at $(\bar{x}, \bar{y})$. Then the following assertions hold:*

(i) *For every $\varepsilon > 0$ there exist a suboptimal triple*

$$(x, y, w) \in \prod_{i=1}^{n} \operatorname{cl} P_i(\bar{x}) \times \prod_{j=1}^{m} \operatorname{cl} S_j \times \operatorname{cl} W$$

with w defined in (2.1) *and a common marginal price $p^* \in E^* \setminus \{0\}$ satisfying*

$$-p^* \in \widehat{N}(x_i; \operatorname{cl} P_i(\bar{x})) + \varepsilon I\!B^*, \tag{4.1}$$

with $x_i \in \bar{x}_i + \frac{\varepsilon}{2} I\!B$ for all $i = 1, \ldots, n$,

$$p^* \in \widehat{N}(y_j; \operatorname{cl} S_j) + \varepsilon I\!B^* \tag{4.2}$$

with $y_j \in \bar{y}_j + \frac{\varepsilon}{2} I\!B$ for all $j = 1, \ldots, m$,

$$p^* \in \widehat{N}(w; \operatorname{cl} W) + \varepsilon I\!B^* \tag{4.3}$$

with $w \in \bar{w} + \frac{\varepsilon}{2} I\!B$, and

$$\frac{1 - \varepsilon}{2\sqrt{n + m + 1}} \le \|p^*\| \le \frac{1 + \varepsilon}{2\sqrt{n + m + 1}}, \tag{4.4}$$

where $\bar{w}$ is defined in (2.2).

(ii) *Furthermore, conditions* (4.1)–(4.3) *are equivalent to the existence of real-valued functions g_i, $i = 1, \ldots, n$, and h_j, $j = 1, \ldots, m+1$, on the commodity space E that are Fréchet differentiable at x_i, y_j, and w, respectively, with*

$$\begin{cases} \|\nabla g_i(x_i) - p^*\| \le \varepsilon, \quad i = 1, \ldots, n, \\ \|\nabla h_j(y_j) - p^*\| \le \varepsilon, \quad j = 1, \ldots, m, \\ \|\nabla h_{m+1}(w) - p^*\| \le \varepsilon \end{cases} \tag{4.5}$$

and such that each g_i, $i = 1, \ldots, n$, achieves its global minimum over $\operatorname{cl} P_i(\bar{x})$ at x_i, each h_j, $j = 1, \ldots, m$, achieves its global maximum over $\operatorname{cl} S_j$ at y_j, and h_{j+1} achieves its global maximum over $\operatorname{cl} W$ at w.

Proof. To prove assertion (i) in a parallel way for Pareto and weak Pareto optimal allocations we check, following [26, 22], that $(\bar{x}, \bar{y}, \bar{w})$ is a *locally extremal point* of the system of the set system $\{\Omega_1, \Omega_2\}$ defined by

$$\begin{aligned} \Omega_1 := \ & \prod_{i=1}^{n} \Big[\operatorname{cl} P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon I\!B)\Big] \times \prod_{j=1}^{m} \Big[\operatorname{cl} S_j \cap (\bar{y}_j + \varepsilon I\!B)\Big] \\ & \times \Big[\operatorname{cl} W \cap (\bar{w} + \varepsilon I\!B)\Big] \quad \text{and} \end{aligned} \tag{4.6}$$

$$\Omega_2 := \Big\{(x, y, w) \in X \Big| \ \sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j - w = 0\Big\} \tag{4.7}$$

provided that the NDQ and NDWQ condition holds on the Pareto and weak Pareto case, respectively. Then applying the *extremal principle* of Theorem 3.3 to this system of sets, we arrive at all the conclusions (4.1)–(4.4) with the *common marginal price* p^*.

To justify assertion (ii), we take p^* satisfying the marginal price conditions (4.1)–(4.3) and find p_i^* and p_j^* such that

$$\begin{cases} -p_i^* \in \widehat{N}(x_i; \operatorname{cl} P_i(\bar{x})), \ \ \|p_i^* - p^*\| \le \varepsilon, \quad i = 1, \ldots, n, \\ p_j^* \in \widehat{N}(y_j; \operatorname{cl} S_j), \ \ \|p_j^* - p^*\| \le \varepsilon, \quad j = 1, \ldots, m, \\ p_{j+1}^* \in \widehat{N}(w; \operatorname{cl} W), \ \ \|p_{j+1}^* - p^*\| \le \varepsilon. \end{cases} \tag{4.8}$$

Putting $\nabla g_j(x_i) = -p_i^*$ as $i = 1, \ldots, n$, $\nabla h_j(y_j) = p_j^*$ as $j = 1, \ldots, m$, and $\nabla h_{j+1} = p_{j+1}^*$ and then applying the *smooth variational description* of Fréchet normals from assertion (i) of Theorem 3.2 held in arbitrary Banach spaces, we arrive at all the conclusions in (ii). △

The established conclusions of Theorem 4.1(ii) can be naturally interpreted as follows: using *nonlinear prices* given by the functions g_i and h_j differentiable at suboptimal allocations with the derivatives (i.e., *rates of change*) arbitrarily close to the *linear* marginal price p^*, we are able to *approximately* achieve a certain *decentralized equilibrium* in fully *nonconvex* models that is similar to the classical (maximization-minimization) equilibrium as in the Arrow-Debreu second welfare theorem for convex models of welfare economics. Note the specific representation of ε-normals to convex sets in (3.4) easily leads a specific *perturbed* form of the decentralized equilibrium for convex models; cf. [2, 16, 26, 22].

Additional assumptions on the commodity space of $\mathcal{E}$ allow us to provide *more specified information* on nonlinear prices supporting the approximate decentralized equilibrium.

Theorem 4.2 (refined decentralized nonlinear prices in nonconvex economies). *Let $(\bar{x}, \bar{y})$ be a local Pareto (resp. weak Pareto) optimal allocation of the economy $\mathcal{E}$ under the corresponding assumptions of Theorem* 4.1*, and let p^* be the common marginal price satisfying conditions* (4.1)–(4.4) *at the suboptimal allocation (x, y, w). Then there are nonlinear prices g_i, $i = 1, \ldots, n$, and h_j, $j = 1, \ldots, m+1$, whose rates of change are arbitrarily close to p^* as in* (4.5) *satisfying the following additional properties:*

(i) *Suppose that E admits an equivalent norm Fréchet differentiable off the origin. Then all g_i and h_j can be chosen as Fréchet differentiable on E and such that:*

—*-each g_i is concave on E and achieves the global minimum over $\operatorname{cl} P_i(\bar{x})$ uniquely at x_i for $i = 1, \ldots, n$;*

—*-each h_j is concave on E achieving the global maximum over $\operatorname{cl} S_j$ uniquely at y_i for $j = 1, \ldots, m$ and the global maximum over $\operatorname{cl} W$ uniquely at w for $j = m + 1$.*

(ii) *Suppose that E admits an $\mathcal{S}$-smooth bump function from the classes $\mathcal{S}$ considered in Theorem* 3.2(iii)*. Then all g_i and h_j can be chosen as $\mathcal{S}$-smooth on E and such that:*

—*-each g_i achieves the global minimum over $\operatorname{cl} P_i(\bar{x})$ uniquely at x_i whenever $i = 1, \ldots, n$;*

—*-each h_j achieves the global maximum over $\operatorname{cl} S_j$ uniquely at y_i for $j = 1, \ldots, m$ and the global maximum over $\operatorname{cl} W$ uniquely at w for $j = m + 1$.*

Proof. We proceed as in the proof of Theorem 4.1 employing now the *refined smooth variational descriptions* of Fréchet normals from assertions (ii) and (iii) of Theorem 3.2 in (4.8) instead of that from Theorem 3.2(i). △

Next we provide necessary optimality conditions for Pareto and weak Pareto optimal allocations of the nonconvex economy $\mathcal{E}$ in the *exact/pointbased* form of the extended second welfare theorem under additional *sequential normal compactness* (SNC) assumptions imposed on *either one* of the preference, production, or net demand constraint sets. Note that SNC property is the weakest among known compactness-like requirements needed for exact forms of the second welfare theorem.

In particular, it is generally weaker that the CEL property imposed in the corresponding extensions by Jofré [16], Flam [13], and Florenzano et al. [14]; cf. Section 3. In this way we obtain an improvement of even the classical second welfare theorem for convex economies; see below.

Following the previous pattern of Theorems 4.1 and 4.2, we present results in the two forms: using first *linear marginal equilibrium prices* formalized via our basic normal cone (3.2) and then as a *limiting decentralized equilibrium* in nonconvex models realized via *nonlinear prices*. Observe some similarity between this limiting decentralized equilibrium with nonlinear prices and the so-called "virtual equilibrium" introduced recently by Jofré, Rockafellar and Wets [18] in convex Walrasian models of exchange via a limiting procedure from a classical equilibrium.

Theorem 4.3 (exact forms of the extended second welfare theorem for Pareto and weak Pareto optimal allocations). *Let $(\bar{x}, \bar{y})$ be a local Pareto (resp. weak Pareto) optimal allocation of the economy $\mathcal{E}$ satisfying the corresponding assumptions of Theorem 4.1 with $\bar{w}$ defined in (2.2). Assume also that one of the sets*

$$\operatorname{cl} P_i(\bar{x}), \quad i = 1, \ldots, n; \quad \operatorname{cl} S_j, \quad j = 1, \ldots, m; \quad \operatorname{cl} W$$

is sequentially normally compact at $\bar{x}_i$, $\bar{y}_j$, and $\bar{w}$, respectively. Then the following hold:

(i) *There is a marginal equilibrium price $p^* \in E^* \setminus \{0\}$ satisfying*

$$-p^* \in N(\bar{x}_i; \operatorname{cl} P_i(\bar{x})), \quad i = 1, \ldots, n, \tag{4.9}$$

$$p^* \in N(\bar{y}_j; \operatorname{cl} S_j), \quad j = 1, \ldots, m, \tag{4.10}$$

$$p^* \in N(\bar{w}; \operatorname{cl} W). \tag{4.11}$$

(ii) *Under the assumptions made, there exist sequences of smooth nonlinear prices $g^k = (g_1^k, \ldots, g_n^k)$ and $h^k = (h_1^k, \ldots, h_{m+1}^k)$ and sequences of suboptimal allocations*

$$(x^k, y^k) \in \prod_{i=1}^{n} \operatorname{cl} P_i(\bar{x}) \times \prod_{j=1}^{m} \operatorname{cl} S_j \quad \textit{with} \quad w^k := \sum_{i=1}^{n} x_i^k - \sum_{j=1}^{m} y_j^k \in \operatorname{cl} W$$

such that $(x^k, y^k, w^k, g^k, h^k)$ forms a nonlinear price decentralized equilibrium, which arbitrarily closely approximates the marginal price equilibrium $(\bar{x}, \bar{y}, \bar{w}, p^*)$. The latter means that:

—– for all $k \in I\!N$, each g_i^k and h_j^k is Fréchet differentiable at x_i^k, y_j^k, and w^k achieving the global minimum over $\operatorname{cl} P_i(\bar{x})$ and the global maximum over $\operatorname{cl} S$ and $\operatorname{cl} W$ at these points for $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $j = m + 1$, respectively;

—–one has the convergence

$$(x^k, y^k, w^k) \to (\bar{x}, \bar{y}, \bar{w}), \quad \nabla g_i^k(x_i^k) =: p_i^k \xrightarrow{w^*} p^*, \quad \nabla h_j^k(y_j^k) =: p_j^k \xrightarrow{w^*} p^*, \quad \nabla h_{m+1}^k(w^k) =: p_{m+1}^k \xrightarrow{w^*} p^*$$

as $k \to \infty$ with $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Furthermore, under the additional assumptions on the commodity space E made in assertions (i) *and* (ii) *of Theorem 4.2, the approximate nonlinear prices g^k and h^k enjoy the corresponding properties listed therein for all $k \in I\!N$.*

Proof. To justify assertion (i), we proceed similarly to the proof of [26, Theorem 4.4] and [22, Theorem 5.2] by passing to the limit from the relationships of Theorem 4.1(i) as $\varepsilon \downarrow 0$.

To establish the first part of assertion (ii), take $\varepsilon := 1/k$, $k \in I\!N$, in Theorem 4.1 and denote by (x^k, y^k, w^k) the sequence of suboptimal allocations and by (p^k, g^k, h^k) the sequence of the corresponding linear and nonlinear prices satisfying all the conclusions in both assertions (i) and (ii) of that theorem for $\varepsilon = 1/k$. It follows from the mentioned proof of assertion (i) of this theorem that $p^k \stackrel{w^*}{\to} p^* \neq 0$ as $k \to \infty$, where p^* is a marginal price satisfying (4.9)–(4.11). Furthermore, it follows from relations (4.5) as $\varepsilon = 1/k$ that

$$\|p_i^k - p^k\| \to 0, \;\; i = 1, \ldots, n, \;\; \text{and} \;\; \|p_j^k - p^k\| \to 0, \;\; j = 1, \ldots, m+1, \quad \text{as} \;\; k \to \infty,$$

where $p_i^k := \nabla g_i^k(x_k)$ for $i = 1, \ldots, n$, $p_j^k := \nabla h_j^k(y_j^k)$ for $j = 1, \ldots, m$, and $p_{m+1}^k := \nabla h_{m+1}^k(w^k)$. This justifies the weak* convergence of all p_i^k and p_j^k to p^* as claimed in the theorem. The last statement of the theorem follows from this procedure by applying assertions (i) and (ii) of Theorem 4.2 at each approximating step with $\varepsilon = 1/k$. $\triangle$

Note that the limiting *decentralized* descriptions of the marginal price equilibrium from Theorem 4.3 are achieved due to the *refined formalization of marginal prices* in the extended second welfare theorem via our nonconvex *basic normal cone*. It does not seem to be possible to derive results of this type from previous formalizations of marginal prices in nonconvex models of welfare economics via the Clarke normal cone and also via Ioffe's extensions of the basic normal cone to the general Banach space setting; cf. [15, 19, 29, 30] for more details and discussions.

For models of welfare economics with *convex* data, we arrive at the following corollary of Theorem 4.3, which provides an improvement of the classical second welfare theorem for convex economies with Asplund commodity spaces for Pareto and weak Pareto optimal allocations.

Corollary 4.4 (improved second welfare theorem for convex economies). *In addition to the assumptions of Theorem* 4.3*, suppose that all the preference and production sets*

$$P_i(\bar{x}), \;\; i = 1, \ldots, n, \quad \textit{and} \quad S_j, \;\; j = 1, \ldots, m,$$

are convex and that the net demand constraint set W *admits the representation*

$$W = \omega + \Gamma \;\; \textit{with some} \;\; \omega \in \operatorname{cl} W,$$

where Γ *is a nonempty convex subcone of* E*. Then there is a nonzero price* $p^* \in E^*$ *satisfying the vanishing excess demand condition*

$$\Big\langle p^*, \sum_{i=1}^n \bar{x}_i - \sum_{j=1}^m \bar{y}_j - \omega \Big\rangle = 0 \tag{4.12}$$

and the decentralized equilibrium relationships

$$\left\{ \begin{array}{l} \bar{x}_i \;\; \textit{minimizes} \;\; \langle p^*, x_i \rangle \;\; \textit{over} \;\; x_i \in \operatorname{cl} P_i(\bar{x}_i) \;\; \textit{whenever} \;\; i = 1, \ldots, n, \\ \bar{y}_j \;\; \textit{maximizes} \;\; \langle p^*, y_j \rangle \;\; \textit{over} \;\; y_j \in \operatorname{cl} S_j \;\; \textit{whenever} \;\; j = 1, \ldots, m. \end{array} \right. \tag{4.13}$$

Note that the assumptions of Corollary 4.4 essentially improve the *nonempty interiority* condition of the classical second welfare theorem for convex economies. As shown and discussed in [12, 29], there are *convex sets in Asplund spaces, which are SNC while having empty even relative interiors and not being CEL.*

5 Nonconvex Economies with Ordered Commodity Spaces

In the concluding section of the paper we study our basic model $\mathcal{E}$ of welfare economics whose commodity space E is *ordered* by the *closed positive cone*

$$E_+ := \{e \in E|\ e \geq 0\},$$

where the (standard) partial ordering relation is denoted by $\geq$, in accordance with the conventional notation in the economic literature. Note that in what follows we *do not* impose a *lattice structure* on E and *do not* assume the fulfillment of any *properness* condition of the Mas-Colell type.

The first result provide a specification of Theorem 4.3 for the case of ordered commodity spaces ensuring the *marginal price positivity* by [22, Proposition 4.7].

Theorem 5.1 (price positivity for Pareto and weak Pareto optimal allocations). *Let* $(\bar{x}, \bar{y})$ *be a local Pareto* (*resp. weak Pareto*) *optimal allocation of the economy* $\mathcal{E}$*. In addition to the corresponding assumptions of Theorem* 4.3*, suppose that* E *is an ordered space and that* ONE *of the following conditions holds:*

(a) *There is a consumer index* $i \in \{1, \ldots, n\}$ *such that the corresponding consumer satisfies the* DESIRABILITY CONDITION *at* $\bar{x}$*:*

$$\operatorname{cl} P_i(\bar{x}) + E_+ \subset \operatorname{cl} P_i(\bar{x}). \tag{5.1}$$

(b) *There is a production* $j \in \{1, \ldots, m\}$ *such that the corresponding firm satisfies the* FREE DISPOSAL *condition:*

$$\operatorname{cl} S_j - E_+ \subset \operatorname{cl} S_j. \tag{5.2}$$

(c) *The net demand constraint set* W *exhibits the* IMPLICIT FREE DISPOSAL *of commodities:*

$$\operatorname{cl} W - E_+ \subset \operatorname{cl} W. \tag{5.3}$$

Then there is a positive marginal price $p^* \in E^*_+ \setminus \{0\}$ *satisfying relations* (4.9)–(4.11)*. Furthermore, the marginal price equilibrium* $(\bar{x}, \bar{y}, \bar{w}, p^*)$ *can be arbitrarily closely approximated by a sequence of nonlinear price decentralized equilibria* $(x^k, y^k, w^k, q^k, h^k)$ *as in assertion* (ii) *of Theorem* 4.3*.*

Observe that each of the conditions in (a)–(c) implies the *epi-Lipschitzian* property of the corresponding sets $\operatorname{cl} P_i(\bar{x})$, $\operatorname{cl} S_j$, and $\operatorname{cl} W$ *provided* that $\operatorname{int} E_+ \neq \emptyset$. It is not hard to check that the latter *nonempty interior* requirement on the *positive cone* of E ensures also the fulfillment of the qualification and normal compactness conditions of Theorem 4.3 and thus the existence of a *positive marginal price* $p^* \in E^*_+ \setminus \{0\}$ and the approximating nonlinear prices of decentralized equilibria in Theorem 5.1 for Pareto and weak Pareto optimal allocations.

Our principal observation is that the above net demand qualification conditions, related to the nonempty interiority requirement $\operatorname{int} E_+ \neq \emptyset$ for ordered commodity spaces, are *not needed at all for strong Pareto* optimal allocations of convex and nonconvex economies, where $\operatorname{int} E_+ = \emptyset$ in many settings important for both the theory and applications. We present approximate and exact versions of the extended second welfare theorem for strong Pareto optimal allocations. Some of the results below require the *generating condition* on the closed positive cone $E_+ \subset E$, which means that $E_+ - E_+ = E$. This class of Banach spaces is sufficiently large including, in particular, all *Riesz spaces* whose generating positive cones typically have empty interiors.

Theorem 5.2 (approximate marginal and decentralized forms of the extended second welfare theorem for strong Pareto optimal allocations). *Let* $(\bar{x}, \bar{y})$ *be a local strong Pareto optimal allocation of the economy* $\mathcal{E}$ *with an ordered Asplund commodity space* E*, and let the sets* S_j*,* W *be locally closed near* $\bar{y}_j$ *and* $\bar{w}$*, respectively. Then the following hold:*

(i) *Assume that the closed positive cone* E_+ *is generating and that either the economy exhibits the implicit free disposal of commodities*

$$W - E_+ \subset W, \tag{5.4}$$

or the free disposal production condition

$$S_j - E_+ \subset S_j \ \ \text{for some} \ \ j \in \{1, \ldots, m\} \tag{5.5}$$

is fulfilled, or $n > 1$ *and there is a consumer* $i_0 \in \{1, \ldots, n\}$ *such that* $P_{i_0}(\bar{x}) \neq \emptyset$ *and one has the desirability condition*

$$\operatorname{cl} P_i(\bar{x}) + E_+ \subset \operatorname{cl} P_i(\bar{x}) \ \ \text{for some} \ \ i \in \{1, \ldots, n\} \setminus \{i_0\} \tag{5.6}$$

Then for every $\varepsilon > 0$ *there exist a suboptimal triple*

$$\begin{aligned} (x, y, w) \ \ &\in \prod_{i=1}^{n} \Big[\operatorname{cl} P_i(\bar{x}) \bigcap \Big(\bar{x}_i + \frac{\varepsilon}{2} I\!B \Big) \Big] \\ &\times \prod_{j=1}^{m} \Big[S_j \bigcap \Big(\bar{y}_j + \frac{\varepsilon}{2} I\!B \Big) \Big] \times \Big[W \bigcap \Big(\bar{w} + \frac{\varepsilon}{2} I\!B \Big) \Big] \end{aligned} \tag{5.7}$$

with w *from* (3.1) *and an common marginal price* $p^* \in E^*$ *satisfying relations* (4.1)–(4.4)*.*

(ii) *Under the assumptions made in* (i)*, there exist real-valued functions* g_i*,* $i = 1, \ldots, n$*, and* h_j*,* $j = 1, \ldots, m+1$*, on* E *that are Fréchet differentiable at* x_i*,* y_j*, and* w*, respectively, satisfy* (4.5)*, and such that each* g_i*,* $i = 1, \ldots, n$*, achieves its global minimum over* $\operatorname{cl} P_i(\bar{x})$ *at* x_i*, each* h_j*,* $j = 1, \ldots, m$*, achieves its global maximum over* S_j *at* y_j*, and* h_{j+1} *achieves its global maximum over* W *at* w*. Furthermore, the nonlinear prices* g_i *and* h_j *possess the additional properties listed in Theorem* 4.2 *under the corresponding assumptions made therein.*

(iii) *All the conclusions in* (i) *and* (ii) *hold true if, instead of the assumption that* E_+ *is a generating cone, we suppose that* $E_+ \neq \{0\}$ *and at least two among the sets* W*,* S_j *for* $j = 1, \ldots, m$*, and* $P_i(\bar{x})$ *for* $i = 1, \ldots, n$ *satisfy the corresponding conditions in* (5.4)–(5.6)*.*

Proof. Consider the system of two sets $\{\Omega_1, \Omega_2\}$ defined in (4.6), where the closure operation for S_j and W in the construction of Ω_2 is omitted, since these sets are locally closed around the points of interest. Taking a *strong* Pareto local optimum $(\bar{x}, \bar{y})$ of $\mathcal{E}$, check by following the proof of [26, Theorem 5.5] and using the fundamental Krein-Šmulian theorem [1] that $(\bar{x}, \bar{y}, \bar{w}) \in \Omega_1 \cap \Omega_2$ is a *locally extremal point* of $\{\Omega_1, \Omega_2\}$ if either the assumptions in (i) or those in (iii) hold.

Applying now the extremal principle of Theorem 3.3 to system (4.6), we find a suboptimal allocation (x, y, w) satisfying (5.7) and an approximate marginal price p^* satisfying (4.1)–(4.4). The nonlinear price conclusions formulated in (ii) are proved similarly to those in Theorems 4.1(ii) and 4.2 under the new assumptions made in the case of strong Pareto optimal allocations. △

Our next results establish strong Pareto versions of the exact second welfare theorem in both terms of (*positive*) *marginal price* and *limiting decentralized nonlinear price equilibria*.

Theorem 5.3 (exact forms of the extended second welfare theorem for strong Pareto optimal allocations). *Let $(\bar{x}, \bar{y})$ be a local strong Pareto optimal allocation of the economy $\mathcal{E}$ with an ordered Asplund commodity space E, and let the sets S_j, W be locally closed near $\bar{y}_j$ and $\bar{w}$, respectively. Suppose in addition to the assumptions made in either* (i) *or* (ii) *of Theorem* 5.2 *that one of the sets*

$$\mathrm{cl}\, P_i(\bar{x}),\ \ i = 1, \ldots, n, \qquad S_j,\ \ j = 1, \ldots, m, \qquad W$$

is SNC at the corresponding points. Then the following hold:

(i) *There is a positive marginal price $p^* \in E^* \setminus \{0\}$ satisfying the pointbased relationships* (4.9)–(4.11) *formalized via the basic normal cone.*

(ii) *There exist sequences of smooth nonlinear prices $g^k = (g_1^k, \ldots, g_n^k)$ and $h^k = (h_1^k, \ldots, h_{m+1}^k)$ and sequences of suboptimal allocations*

$$(x^k, y^k) \in \prod_{i=1}^n \mathrm{cl}\, P_i(\bar{x}) \times \prod_{j=1}^m \mathrm{cl}\, S_j \ \ \text{with} \ \ w^k := \sum_{i=1}^n x_i^k - \sum_{j=1}^m y_j^k \in \mathrm{cl}\, W$$

such that $(x^k, y^k, w^k, g^k, h^k)$ forms a nonlinear price decentralized equilibrium, which arbitrarily closely approximates the (*positive*) *marginal price equilibrium $(\bar{x}, \bar{y}, \bar{w}, p^*)$ in the sense of Theorem* 4.3*, with the additional specifications of the nonlinear prices listed therein.*

Proof. To establish the marginal price relationships in assertion (i), we pass to the limit in those from assertion (i) of Theorem 5.2 as $\varepsilon \downarrow 0$. The positivity of the limiting marginal price p^* follows from (4.9)–(4.11) under either one of the conditions (5.4)–(5.6), as in the proof of Theorem 5.1. Assertion (ii) about the limiting nonlinear price decentralized equilibrium is proved similarly to the one in Theorem 4.3(ii). $\triangle$

A counterpart of assertion (i) of Theorem 5.3 was obtained by Khan [19] in terms of Ioffe's normal cone in general Banach spaces under substantially more restrictive assumptions. Note that the latter cone appeared as another infinite-dimensional extension of our basic normal cone (3.3) being generally larger (never smaller) than the basic normal cone (3.2), even in the Asplund space setting. We refer the reader to the paper by Mordukhovich [26] and the book [29] for *abstract* analogs of some results presented above that are formulated via *axiomatically defined* prenormal and normal structures on arbitrary Banach spaces. Let us emphasize that such abstract analogs concern only some (not all) results of the *marginal price* type and do not relate to *nonlinear price decentralized equilibria*, where the usage of the basic normal cone seems to be crucial.

To conclude this paper, we mention the possibility of extending the methods and results of this paper to welfare economic models with *public goods*. In contrast to the welfare economic model studied above, economies with *public goods* involve two categories of commodities: private and public. Mathematically this means that the commodity space E is represented as the product of two Banach spaces $E = X \times Z$, where X and Z are the space of private and public commodities, respectively. Thus consumer variables $x_i \in X$, $i = 1, \ldots, n$, stand for private goods, while those of $z_i \in Z$, $i = 1, \ldots, n$, correspond to public goods of commodities; $y_j \in S_j \subset E$ connote production variables as above. Considering for simplicity the "markets clear" setting in (2.1) with the given initial endowment of scare recourses $\omega \in X$ only for *private* goods, we write the market constraints in the economy involving both private and public goods as follows:

$$\sum_{i=1}^n (x_i, z_i) - \sum_{j=1}^m y_j = (\omega, 0). \tag{5.8}$$

Note that the market constraint condition (5.8) reflects the fact that there is *no endowment of public goods*, which is the most crucial characteristic feature of public good economies.

The main changes for public good economies, in comparison with the above marginal price results of this paper, are as follows presented only for the case of the exact/limiting conditions from in Theorem 4.3: instead of the existence of a nonzero marginal price $p^* \in E^*$ satisfying (4.9) and (4.10), we have prices $p^* = (p_x^*, p_z^*) \in X^* \times Z^*$ and $p_i^* \in Z^*$ as $i = 1, \ldots, n$ with $(p_x^*, p_i^*) \neq 0$ for at least one $i \in \{1, \ldots, n\}$ and such that

$$-(p_x^*, p_z^*) \in N(\bar{x}_i; \operatorname{cl} P_i(\bar{x})), \quad i = 1, \ldots, n, \tag{5.9}$$

$$(p_x^*, p_z^*) \in N(\bar{y}_j; \operatorname{cl} S_j), \quad j = 1, \ldots, m, \quad \text{and} \tag{5.10}$$

$$p_z^* = \sum_{i=1}^{n} p_i^*. \tag{5.11}$$

Observe that, while conditions (5.9) and (5.10) are actually concretizations of those in (4.9) and (4.10) for the product structure of the commodity space $E = X \times Z$, the last one in (5.11) confirms the fundamental conclusion for welfare economics with public goods that goes back to Samuelson [33]: *the marginal rates of transformation for public goods are equal to the sum of the individual marginal rates of substitution* at Pareto optimal allocations.

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