Theory and applications of convex and non-convex feasibility problems

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Abstract. A *feasibility problem* requests solution to the problem

$$\text{Find } \ x \in \bigcap_{i=1}^{N} C_i$$

where $C_1, C_2, \ldots C_N$ are closed sets finitely many closed sets lying in a Hilbert space $\mathcal{H}$.

We consider *iterative methods* based on the non-expansive properties of the *metric projection* operator

$$P_C(x) := \operatorname{argmin}_{c \in C} \|x - c\|$$

or *reflection operator* $R_C := 2P_C - I$ on a closed convex set C in Hilbert space. These methods work best when the projection on each set C_i is easy to describe or approximate.

These methods are especially useful when the number of sets involved is large as the methods are fairly easy to parallelize. The theory is pretty well understood when all sets are convex. The theory is much less clear in the non-convex case. But as we shall see application of this case has had may successes. So this is a fertile area for both pure and applied study.

The five hours of lectures will cover the following topics.

1. *Alternating projection methods*: background theory, convergence and basic algorithms ([6, 5, 17], [8] and [14, 13])

2. *The Douglas Rachford reflection method* and generalizations ([9, 1, 10, 11, 4], [15] and [16])

3. *Applications* to convex problems and to non-convex *combinatorial problems* [2] and to *matrix completion problems* [3]

4. *Protein conformation determination*: a detailed case study [3]

5. *Relaxed reflection methods* and norm convergence for realistic problems [12]

This is based on joint work with Matt Tam, Brailey Sims and Fran Aragon.

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