On some universal Morse–Sard type theorems and applications in fluid mechanics

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The next general problem is a kind of synthesis of the Morse-Sard type theorems and the Luzin N-property for Hausdorff measures: *Let* $f:\mathbb{R}^n\to\mathbb{R}^d$ *be some regular function,* S *be a subset of critical set* $\{\operatorname{rank}\nabla f\le m\}$*, and let the equality* $\mathscr{H}^\tau(S)=0$ *(or the inequality* $\mathscr{H}^\tau(S)<\infty$ *) hold for some* $\tau>0$*. Does it imply that* $\mathscr{H}^\sigma(f(E))=0$ *for some* $\sigma=\sigma(\tau,m)$*?*

For classical C^k-smooth and Hölder classes of mappings this problem was solved by Bates S.M. and Moreira C. [2, 12]. For Sobolev classes of mappings this problem was completely solved in the recent papers [5]–[7], which include also the corresponding Fubiny type properties (á la Dubovitskiĭ–Sard theorems). The results of [5]–[7] covered also the cases of fractional Sobolev and Sobolev–Lorentz spaces, and they really summarize many years of research started in our joint papers with Jean Bourgain and Jan Kristensen [3, 4].

There are many surprising phenomena in these extensions of Morse–Sard properties to Sobolev functions, for example, under the limiting regularity assumptions almost all level sets of these functions turns out to be classically smooth manifolds despite the fact that functions itself are not smooth — in general they are continuous only.

These results were very helpful for the solution of the so-called Leray's problem in mathematical fluid mechanics. The problem remained open for more than 80 years (starting from the publication of the famous paper by Jean Leray 1933 [11]). Namely, for plane and axially symmetric spatial flows the existence theorem was proved [8] for boundary value problem of stationary Navier-Stokes equations in bounded domains under necessary and sufficient condition of zero total flux (see also [9] for the case of exterior 3D domains).

The last lectures address to solutions of stationary Navier–Stokes system in two dimensional exterior (unbounded) domains, namely, existence of these solutions and their asymptotic behavior. The problem is still open (almost 90 years!) and considered as one of the most challenging. The talks are based on our recent joint papers (see, e.g., [10]), where the uniform boundedness and uniform convergence at infinity for arbitrary solution with finite Dirichlet integral were established. Surprisingly, the Real Analysis tools (especially the Coarea formula) turned out to be very effective here. In the proofs we develop also the ideas of the classical paper by Amick [1].

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