Capacitary Integration and Sobolev Inequalities

Daniel Spector

National Taiwan Normal University, Taiwan

Sobolev spaces emerge almost immediately in the modeling of physical phenomena via partial differential equation or through minimization of energies in the calculus of variations. These spaces are the functional completion of smooth functions in various norms, and are useful in both giving meaning to equations and energies where functions are not known to possess sufficient differentiability properties to express these notions classically, as well as to obtain existence of solutions or minimizers via the compactness properties of these spaces. The concept of capacity is intrinsic to Sobolev spaces, and therefore to the study of nature in the modern paradigm. The aim of this series of lectures is to present some results concerning capacities that the lecturer finds of interest in order to motivate their further study. The first several lectures concern the emergence of capacity, its use as a tool in the study of Sobolev spaces, and the strong-type capacitary Sobolev inequalities of Adams, Maz'ya-Adams-Dahlberg-Hansson, Maz'ya-Meyers-Ziemer, and Korobkov-Kristensen. While these results are classical, we feel that our organization and emphasis of a few key ideas might be useful to encourage the student to undertake a more detailed study of the literature, e.g. Adams and Hedberg's Function Spaces and Potential Theory, Ziemer's Weakly Differentiable Functions, Maly and Ziemer's Fine Regularity of Solutions of Elliptic Partial Differential Equations, and Evans and Gariepy's Measure Theory and Fine Properties of Functions. The concluding lectures concern more recent work on capacities, including strong-type capacitary Sobolev inequalities in L^1, capacitary BMO spaces, and capacitary maximal estimates and their application. These results have been obtained in the last several years and have a number of open questions that have not been settled which we will present in the lectures.