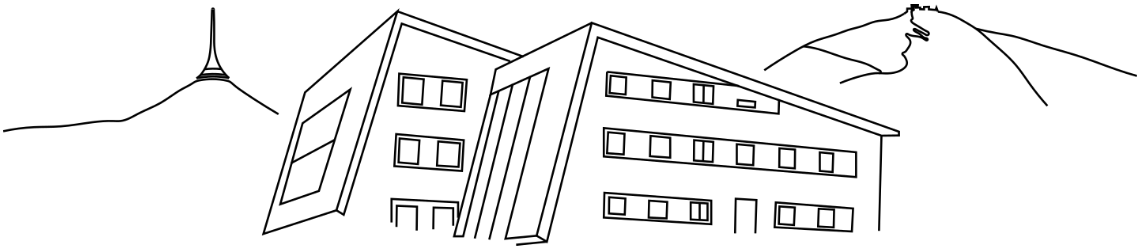
Spring School on Analysis 2023

Function Spaces and Applications XII



Book of Abstracts

FACULTY OF ELECTRICAL ENGINEERING CTU IN PRAGUE

FACULTY OF MATHEMATICS AND PHYSICS Charles University

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Invited lecture courses

Petru Mironescu
Université Claude Bernard Lyon 1
Sobolev maps to manifolds
Abstract: Sobolev spaces $W^{s,p}$ of maps with values into a compact manifold naturally appear in geometry and material sciences. They exhibit qualitatively different properties from scalar Sobolev spaces: in general, there is no density of smooth maps, and standard trace theory fails. We will present some of their basic properties, with focus on the case $0 < s < 1$, where harmonic analysis tools combined with geometric considerations are quite effective. More specifically, we will address questions concerning the existence of topological invariants, of pullback of forms, the density of nice maps, existence of lifting and extensions. Another topic we discuss is the factorization of unimodular maps, which can be seen as a geometric version of paraproducts.

Daniel Spector
National Taiwan Normal University, Taipei City
Capacitary Integration and Sobolev Inequalities
Abstract: Sobolev spaces emerge almost immediately in the modeling of physical phenomena via partial differential equation or through minimization of energies in the calculus of variations. These spaces are the functional completion of smooth functions in various norms, and are useful in both giving meaning to equations and energies where functions are not known to possess sufficient differentiability properties to express these notions classically, as well as to obtain existence of solutions or minimizers via the compactness properties of these spaces. The concept of capacity is intrinsic to Sobolev spaces, and therefore to the study of nature in the modern paradigm. The aim of this series of lectures is to present some results concerning capacities that the lecturer finds of interest in order to motivate their further study. The first several lectures concern the emergence of capacity, its use as a tool in the study of Sobolev spaces, and the strong-type capacitary Sobolev inequalities of Adams, Maz'ya-Adams-Dahlberg-Hansson, Maz'ya-Meyers-Ziemer, and Korobkov-Kristensen. While these results are classical, we feel that our organization and emphasis of a few key ideas might be useful to encourage the student to undertake a more detailed study of the literature, e.g. Adams and Hedberg's Function Spaces and Potential Theory, Ziemer's Weakly Differentiable Functions, Maly and Ziemer's Fine Regularity of Solutions of Elliptic Partial Differential Equations, and Evans and Gariepy's Measure Theory and Fine Properties of Functions. The concluding lectures concern more recent work on capacities, including strong-type capacitary Sobolev inequalities in L^1, capacitary BMO spaces, and capacitary maximal estimates and their application. These results have been obtained in the last several years and have a number of open questions that have not been settled which we will present in the lectures.

Jean Van Schaftingen
Université catholique de Louvain

Limiting Sobolev estimates for vector fields and cancelling differential operators

Abstract: I will present Sobolev-Gagliardo-Nirenberg endpoint estimates for classes of homogeneous vector differential operators. Away of the endpoint cases, the classical Calderón-Zygmund estimates show that the ellipticity is necessary and sufficient to control all the derivatives of the vector field. In the endpoint case, Ornstein has showed that there is no nontrivial estimate on same-order derivatives and the ellipticity is necessary for endpoint Sobolev estimates. Such endpoint estimates were proved first for the deformation operator (Korn-Sobolev inequality by M.J. Strauss) and for the Hodge complex (Bourgain and Brezis). The class of operators for which estimates holds can be characterized by a cancelling condition. The estimates rely on a duality estimate for L^1 vector fields satisfying some conditions on the derivatives, combined with classical algebraic and harmonic analysis techniques. This characterisation unifies classes of known inequalities and extends to the case of Hardy inequalities.

Short talks

Andrea Cianchi
Università di Firenze
Distortion of Hausdorff measures under Orlicz-Sobolev maps
Abstract: A comprehensive theory of the effect of Orlicz-Sobolev maps, between Euclidean spaces, on subsets with zero or finite Hausdorff measure is offered. Arbitrary Orlicz-Sobolev spaces embedded into the space of continuous function and Hausdorff measures built upon general gauge functions are included in our discussion. An explicit formula for the distortion of the relevant gauge function under the action of these maps is exhibited in terms of the Young function defining the Orlicz-Sobolev space. New phenomena and features, related to the flexibility in the definition of the degree of integrability of weak derivatives of maps and the notion of measure of sets, are detected. Classical results, dealing with standard Sobolev spaces and Hausdorff measures, are recovered, and their optimality is shown to hold in a refined stronger sense. Special instances available in the literature, concerning Young functions and gauge functions of non-power type, are also reproduced and, when not sharp, improved. This is joint work with M.V.Korobkov and J.Kristensen.

Alejandro Claros
Basque Center for Applied Mathematics, Bilbao
Self-improving of Poincaré-Sobolev type inequalities
Abstract: In this short talk we will discuss some recent results concerning self-improving of Poincaré-Sobolev inequalities. We obtain those inequalities as a consequence of a general self-improving property shared by functions satisfying the generalized Poincaré inequality $\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq a(Q)$ for all cubes Q, where $f_Q = \frac{1}{|Q|}\int_Q f$ and a is some functional over cubes that satisfying the $SD_p^s(w)$ condition. This is part of a joint work with Carlos Pérez.

Ladislav Drážný
Charles University, Prague
Optimal function spaces in weighted Sobolev embeddings with monomial weight
Abstract: In this short talk we present a Sobolev-type inequality for functions from a certain Sobolev-type space that is built upon a rearrangement-invariant space. Considered rearrangement-invariant spaces are defined on the space $\mathbb{R}^n$ endowed with the measure that is given by a monomial weight. We outline the proof of a so-called reduction principle for the inequality. The reduction principle represents a method of how to characterize the spaces that satisfy the Sobolev-type inequality using one-dimensional inequalities. We also present some results in optimality. For a fixed rearrangement-invariant space, we describe the optimal, i.e. the smallest space such that the Sobolev-type inequality holds. Finally, we deal with some examples. We show that the optimal spaces for concrete Lorentz-Karamata spaces are again Lorentz-Karamata spaces.

Michał Dymek
Warsaw University of Technology
Compactness in the mixed Lebesgue spaces
Abstract: My talk is devoted to the compactness in variable exponent function spaces called the mixed Lebesgue spaces. The theory of variable exponent spaces finds many applications, for instance in nonlinear elasticity, electrorheological fluids or image restoration. Mixed Lebesgue may be considered as an extension of classical Lebesgue spaces. Moreover, the titular spaces play a crucial role in the definition of known variable exponent Besov spaces.

In the classical Lebesgue spaces, relatively compact sets are chacterized by the Riesz-Kolmogorov Theorem. It says that subset of Lebesgue space is precompact if and only if certain three conditions are satisfied. It was observed by Sudakov that one condition in the Riesz-Kolmogorov Theorem is redundant. Therefore, the characterization of precompactness using merely two conditions can be given. In my talk I shall present generalization of Riesz-Kolmogorov Theorem for mixed Lebesgue spaces. Furthermore, I will formulate the Sudakov-type Theorem in the setting of mixed Lebesgue spaces.

Alejandro Santacruz Hidalgo
University of Western Ontario, London
Hardy operators and monotonicity in general spaces
Abstract: An ordered measure space is defined as a measure space together with a totally ordered subset of its sigma algebra called an *ordered core*. Recently, this construction was used in the context of Hardy inequalities, giving a uniform treatment of many different types of Hardy operators. We will begin by introducing a definition of monotone functions compatible with the ordered core and as an application of this framework we will show a weight characterization for two-weight Hardy inequalities which hold on general metric measure spaces. These abstract Hardy operators are closely related to a family of function spaces called down spaces, a variant of the Köthe dual restricted to positive decreasing functions. We will find strong connections to the real-line case; for instance, the down spaces corresponding to L^1 and L^∞ form an exact Calderón-Mityagin couple and as a consequence we can describe all their exact interpolation spaces in terms of the K-functional. This is a joint work with Gord Sinnamon.

Amiran Gogatishvili
Institute of Mathematics CAS, Prague
The duality and equivalents theorems for general Real Interpolation methods
Abstract: In my talk, I will discuss of generalization of the classical Lions-Peetre real interpolation method (K and J) using a general function norm. We will show the relation between J and K in methods and give general formulas for duality. We give applications for general function parameters.

Manvi Grover
Charles University, Prague
Duality results for limiting interpolation spaces
Abstract: We establish conditions under which K and $J-$spaces in the limiting real interpolation involving slowly varying functions have dual spaces. Further, we establish density theorems in limiting interpolation spaces.

Iker Gardeazabal Gutiérrez
Basque Center for Applied Mathematics, Bilbao
Self-improving properties of generalized Poincaré inequalities
Abstract: In this talk we will discuss a geometrical method to obtain extensions of the classical Poincaré-Sobolev inequalities. These results can be seen as applications of the self-improving property of generalized Poincaré inequalities, which will be the content of the first part of the talk. In the second part we will see how to use these results to obtain higher order derivative and fractional Poincaré-Sobolev type inequalities.

Jan Lang
The Ohio State University, Columbus; Charles University, Prague
Sobolev embedding and quality of its non-compactness
Abstract: In this review talk we will look at some non-compact Sobolev embeddings and try to study quality of its non-compactness behavior.

Kai Lüttgen
Technische Universität Chemnitz
Half-Period Cosine Decomposition of Product-Type Function Spaces
Abstract: The half-period cosine basis $\{1\} \cup \{\sqrt{2}\cos(\pi k\cdot) | k \in \mathbb{N}\}$ is one of the many orthonormal bases of $L^2([0,1])$. Recently, Dick, Nuyens and Pillichshammer have studied the Sobolev-type spaces

$$H^s_{\cos}([0,1]) = \left\{ f \in L^2([0,1]) \,\middle|\, \sum_{k\in\mathbb{N}_0} \langle k\rangle^{2s} |\langle f, \cos(\pi k\cdot)\rangle|^2 < \infty \right\}, \quad s > 0,$$

and their relation to the classical Sobolev spaces $W^{m,p}([0,1])$. In this talk we generalise $H^s_{\cos}([0,1])$ to the Besov and Triebel-Lizorkin scales of function spaces. By combining symmetry arguments with a periodisation operator we are able to fill some gaps in the work of Dick et al. Furthermore, in higher dimensions we obtain novel results concerning Besov and Triebel-Lizorkin spaces of product-type, also known as function spaces of dominating mixed smoothness.

Alexandros Matsoukas
National Technical University of Athens
The double phase Dirichlet problem when the lowest exponent is equal to 1
Abstract: Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f \in L^n(\Omega)$. We

consider the following double phase problem

$$\begin{cases} -\operatorname{div}(\frac{\nabla u}{|\nabla u|} + a(x)|\nabla u|^{q-2}\nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega\,. \end{cases}$$

where a is a bounded function with $a(x) \geq 0$ a.e. in Ω.

This is a limiting case of the double phase problem and hence, the solution can be found as the limit of a sequence of solutions of intermediate double phase Dirichlet problems whose lowest exponent p goes to 1. Due to the coexistence of the 1 and the weighted q-Laplacian in the above problem, it is natural to expect that its solution should lie simultaneously in $BV(\Omega)$ and in some suitable weighted Sobolev space.

Our aim in this talk is to discuss an existence and uniqueness result of a solution to this problem, obtained under some specific conditions imposed on the weight function a and the datum.

The talk is based on joint work with Nikos Yannakakis.

Zdeněk Mihula
Czech Technical University in Prague
Different degrees of noncompactness of optimal Sobolev embeddings
Abstract: In this talk, we will take a look at the (subcritical) Sobolev embeddings into L^{p^*} and $L^{p^*,p}$, each of which is in a sense optimal. While both are noncompact and, in a reasonable sense, even so-called maximally noncompact, we will see that there is a substantial difference between how severely noncompact they are. We will express this difference quantitatively in terms of the Bernstein numbers of the embeddings.

Ani Ozbetelashvili
Ivane Javakhishvili Tbilisi State University
$L^{p(\cdot)}$ **Boundedness of the Hardy-Littlewood maximal function on LCA group**
Abstract: In this talk we investigate a boundedness of the Hardy-Littlewood maximal operator M in the variable Lebesgue spaces in the context locally compact abelian group. We show that the local Muckenhoupt condition implies the local boundedness of M.

Dalimil Peša
Charles University, Prague
Wiener–Luxemburg amalgam spaces
Abstract: In this talk we introduce the concept of Wiener–Luxemburg amalgam spaces which are a modification of the more classical Wiener amalgam spaces intended to address some of the shortcomings the latter face in the context of rearrangement-invariant Banach function spaces. We present results concerning many of their properties, including (but nor limited to) their normability, embeddings between them and their associate spaces.

Luboš Pick
Charles University, Prague
Fractional Orlicz–Sobolev spaces
Abstract: We will very briefly survey some recent results on fractional Orlicz–Sobolev spaces. They mainly concern Sobolev-type embeddings of these spaces into the optimal Orlicz target and into the optimal rearrangement-invariant target, embeddings into the space of essentially bounded functions, related Hardy type inequalities, and criteria for compact embeddings. The limits of these spaces when the smoothness parameter $s \in (0, 1)$ tends to either of the endpoints of its range will also be discussed.

This is based on a joint work with Angela Alberico, Andrea Cianchi and Lenka Slavíková.

Artur Słabuszewski
Warsaw University of Technology
Compact embeddings of Hajłasz-Besov and Hajłasz-Triebel-Lizorkin spaces
Abstract: We consider Hajłasz-Besov spaces and Hajłasz-Triebel-Lizorkin spaces defined on a metric-measure space (X, d, μ). During the talk I will present conditions providing compactness of the embeddings into $L^p(X, \nu)$ and $L^q(X, \mu)$ for $q < p$. Moreover, I will show some interesting examples and additional results for measures satisfying δ-doubling condition. The talk is based on joint work with Przemysław Górka and Ryan Alvarado.

Posters / Offprints

Ondřej Bouchala
Charles University, Prague

Existence of quasiconformal mappings in a given Hardy space

Abstract: Let Ω be a simply connected domain in $\mathbb{C}$ and let $0 < p < \infty$. We show that there is a quasiconformal mapping f from the unit disk $\mathbb{D}$ onto Ω which is in the Hardy space H^p. We furthermore show that either all quasiconformal mappings from $\mathbb{D}$ onto Ω are in H^p for every p, or for every $0 < p < \infty$ there is a quasiconformal mapping $f : \mathbb{D} \to \Omega$ with $f \notin H^p$.

Anna Doležalová[1], Stanislav Hencl[1] and Anastasia Molchanova[2]
[1]*Charles University, Prague,* [2]*University of Vienna*

Weak limit of homeomorphisms in $W^{1,n-1}$: invertibility and lower semicontinuity of energy

Abstract: Let Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains and let $f_m : \Omega \to \Omega'$ be a sequence of homeomorphisms with positive Jacobians $J_{f_m} > 0$ a.e. and prescribed Dirichlet boundary data. Let all f_m satisfy the Lusin $\mathbb{N}$ condition and $\sup_m \int_\Omega (|Df_m|^{n-1} + A(|\operatorname{cof} Df_m|) + \phi(J_f)) < \infty$, where A and φ are positive convex functions. Let f be a weak limit of f_m in $W^{1,n-1}$. Provided certain growth behaviour of A and φ, we show that f satisfies the (INV) condition of Conti and De Lellis, the Lusin $\mathbb{N}$ condition, and polyconvex energies are lower semicontinuous.

Anna Doležalová, Stanislav Hencl and Jan Malý
Charles University, Prague

Weak limit of homeomorphisms in $W^{1,n-1}$ and (INV) condition

Abstract: Let $\Omega, \Omega' \subset \mathbb{R}^3$ be Lipschitz domains, let $f_m : \Omega \to \Omega'$ be a sequence of homeomorphisms with prescribed Dirichlet boundary condition and $\sup_m \int_\Omega (|Df_m|^2 + 1/J_{f_m}^2) < \infty$. Let f be a weak limit of f_m in $W^{1,2}$. We show that f is invertible a.e., more precisely it satisfies the (INV) condition of Conti and De Lellis and thus it has all the nice properties of mappings in this class.

Generalization to higher dimensions and an example showing sharpness of the condition $1/J_f^2 \in L^1$ are alsogiven. Using this example we also show that unlike the planar case the class of weak limits and the class of strong limits of $W^{1,2}$ Sobolev homeomorphisms in $\mathbb{R}^3$ are not the same.

Anna Doležalová[1] and Anastasia Molchanova[2]
[1]*Charles University, Prague,* [2]*University of Vienna*

Differentiability almost everywhere of weak limits of bi-Sobolev homeomorphisms

Abstract: This paper investigates the differentiability of weak limits of bi-Sobolev homeomorphisms. Given $p > n - 1$, consider a sequence of homeomorphisms f_k with positive Jacobians $J_{f_k} > 0$ almost everywhere and $\sup_m(||f_k||_{W^{1,n-1}} + ||f_k^{-1}||_{W^{1,p}} < \infty$. We prove that if f and h are weak limits of f_k and f_k^{-1}, re-

spectively, with positive Jacobians $J_f > 0$ and $J_h > 0$ a.e., then $h(f(x)) = x$ and $f(h(y)) = y$ both hold a.e. and f and h are differentiable almost everywhere.

Martin Křepela
Czech Technical University in Prague
Rearrangement-invariant hulls of weighted Lebesgue spaces
Abstract: We characterize the rearrangement-invariant hull, with respect to a given measure μ, of weighted Lebesgue spaces. The solution leads us to first consider when this space is contained in the sum of $(L^1 + L^\infty)(\mathbb{R}, \mu)$ and the final condition is given in terms of embeddings for weighted Lorentz spaces.

Karol Lesnik[1], Tomáš G. Roskovec[2,3] and Filip Soudský[3,4]
[1]*Adam Mickiewicz University in Poznań,* [2]*University of South Bohemia in České Budějovice,* [3]*University of Hradec Králové,* [4]*Technical University of Liberec*
Optimal Gagliardo–Nirenberg interpolation inequality for rearrangement invariant spaces
Abstract: We prove optimality of the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_X \lesssim \|\nabla^2 u\|_Y^{1/2} \|u\|_Z^{1/2},$$

where Y, Z are rearrangement invariant Banach function spaces and $X = Y^{1/2} Z^{1/2}$ is the Calderón–Lozanovskii space. By optimality, we mean that for a certain pair of spaces on the right-hand side, one cannot reduce the space on the left-hand, remaining in the class of rearrangement invariant spaces. The optimality for the Lorentz and Orlicz spaces is given as a consequence, exceeding previous results. We also discuss pointwise inequalities, their importance and counterexample prohibiting an improvement.

Aleš Nekvinda[1] and Hana Turčinová[2]
[1]*Czech Technical University in Prague,* [2]*Charles University, Prague*
Characterization of functions with zero traces via the distance function and Lorentz spaces
Abstract: Consider a regular domain $\Omega \subset \mathbb{R}^N$ and let $d(x) = \operatorname{dist}(x, \partial\Omega)$. Denote $L_a^{1,\infty}(\Omega)$ the space of functions from $L^{1,\infty}(\Omega)$ having absolutely continuous quasinorms. This set is essentially smaller than $L^{1,\infty}(\Omega)$ but, at the same time, essentially larger than a union of all $L^{1,q}(\Omega)$, $q \in [1, \infty)$.

A classical result of late 1980's states that for $p \in (1, \infty)$ and $m \in \mathbb{N}$, u belongs to the Sobolev space $W_0^{m,p}(\Omega)$ if and only if $u/d^m \in L^p(\Omega)$ and $|\nabla^m u| \in L^p(\Omega)$. During the consequent decades, several authors have spent considerable effort in order to relax the characterizing condition. Recently, it was proved that $u \in W_0^{m,p}(\Omega)$ if and only if $u/d^m \in L^1(\Omega)$ and $|\nabla^m u| \in L^p(\Omega)$. In this paper we show that for $N \geq 1$ and $p \in (1, \infty)$ we have $u \in W_0^{1,p}(\Omega)$ if and only if $u/d \in L_a^{1,\infty}(\Omega)$ and $|\nabla u| \in L^p(\Omega)$. Moreover, we present a counterexample which demonstrates that after relaxing the condition $u/d \in L_a^{1,\infty}(\Omega)$ to $u/d \in L^{1,\infty}(\Omega)$ the equivalence no longer holds.

Tomáš G. Roskovec[1] **and Filip Soudský**[2]
[1] *University of South Bohemia in České Budějovice,* [2] *Technical University of Liberec*
An elementary proof of Acerbi Fusco minimizer existence theorem
Abstract: The weak lower semicontinuity of the functional

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx$$

is a classical topic that was studied thoroughly. It was shown that if the function f is continuous and convex in the last variable, the functional is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$. However, the known proofs use advanced instruments of real and functional analysis. Our aim here is to present proof that can be easily understood by students familiar only with the elementary measure theory.

Lenka Slavíková
Charles University, Prague
A sharp variant of the Marcinkiewicz theorem with multipliers in Sobolev spaces of Lorentz type
Abstract: I will exhibit an offprint of my recent joint paper with L. Grafakos and M. Mastyło, in which we establish a sharp variant of the classical Marcinkiewicz theorem on boundedness of Fourier multiplier operators. Our conditions on the symbol of the Fourier multiplier operator are formulated in terms of fractional Sobolev spaces; in particular, we make use of a certain fractional Lorentz-Sobolev space with a logarithmic weight.

Jakub Takáč
University of Warwick
Typical behaviour of 1-Lipschitz images of n-rectifiable metric spaces
Abstract: Suppose X is a complete metric space and E is its $\mathcal{H}^n$-measurable subset. We say that E is n-rectifiable, if it can be covered, up to an $\mathcal{H}^n$-null set, with a countable number of Lipschitz images $g_i(F_i)$ of subsets F_i of $\mathbb{R}^n$. It was recently shown that there is a characterisation of rectifiability in the spirit of Besicovitch-Federer, via the (complete) metric space $(\mathrm{Lip}_1(X, \mathbb{R}^n), \|\cdot\|_\infty)$ of 1-Lipschitz functions $f\colon X \to \mathbb{R}^n$. If E is n-rectifiable, then a typical f in $\mathrm{Lip}_1(X, \mathbb{R}^n)$ satisfies $\mathcal{H}^n(f(E)) > 0$, while if E intersects every n-rectifiable set in an $\mathcal{H}^n$-null set (i.e. E is n-purely unrectifiable), a typical f satisfies $\mathcal{H}^n(f(E)) = 0$. A particular improvement of this result on the rectifiable side is of interest to us, namely, whether, assuming E is rectifiable, there is some $\Delta > 0$ such that a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ (for some fixed $m > n$) satisfies $\mathcal{H}^n(f(E)) > \Delta$. The answer turns out to be complicated. If X is Euclidian then a typical f satisfies even $\mathcal{H}^n(f(E)) = \mathcal{H}^n(E)$ (i.e. $\Delta = \mathcal{H}^n(E)$), but no such $\Delta > 0$ exists for general metric spaces (already for $X = \mathbb{R}^n_\infty$). We discuss sufficient and necessary conditions on the metric space X such that a $\Delta > 0$ having the property described above exists.