Spring School on Analysis 2023: Function Spaces and Applications XII

Offprint Exhibition

Paseky nad Jizerou, 28 May - 3 June 2023

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Existence of quasiconformal mappings in a given Hardy space

Abstract: Let Ω be a simply connected domain in $\mathbb{C}$ and let $0 < p < \infty$. We show that there is a quasiconformal mapping f from the unit disk $\mathbb{D}$ onto Ω which is in the Hardy space H^p. We furthermore show that either all quasiconformal mappings from $\mathbb{D}$ onto Ω are in H^p for every p, or for every $0 < p < \infty$ there is a quasiconformal mapping $f\colon \mathbb{D} \to \Omega$ with $f \notin H^p$.

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Weak limit of homeomorphisms in $W^{1,n-1}$: invertibility and lower semicontinuity of energy

Abstract: Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded domains and let $f_m\colon \Omega \to \Omega'$ be a sequence of homeomorphisms with positive Jacobians $J_{f_m} > 0$ a.e. and prescribed Dirichlet boundary data. Let all f_m satisfy the Lusin $\mathbb{N}$ condition and $\sup_m \int_\Omega (|Df_m|^{n-1} + A(|\operatorname{cof} Df_m|) + \phi(J_f)) < \infty$, where A and φ are positive convex functions. Let f be a weak limit of f_m in $W^{1,n-1}$. Provided certain growth behaviour of A and φ, we show that f satisfies the (INV) condition of Conti and De Lellis, the Lusin $\mathbb{N}$ condition, and polyconvex energies are lower semicontinuous.

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Weak limit of homeomorphisms in $W^{1,n-1}$ and (INV) condition

Abstract: Let $\Omega, \Omega' \subset \mathbb{R}^3$ be Lipschitz domains, let $f_m : \Omega \to \Omega'$ be a sequence of homeomorphisms with prescribed Dirichlet boundary condition and $\sup_m \int_\Omega (|Df_m|^2 + 1/J_{f_m}^2) < \infty$. Let f be a weak limit of f_m in $W^{1,2}$. We show that f is invertible a.e., more precisely it satisfies the (INV) condition of Conti and De Lellis and thus it has all the nice properties of mappings in this class.

Generalization to higher dimensions and an example showing sharpness of the condition $1/J_f^2 \in L^1$ are alsogiven. Using this example we also show that unlike the planar case the class of weak limits and the class of strong limits of $W^{1,2}$ Sobolev homeomorphisms in $\mathbb{R}^3$ are not the same.

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Differentiability almost everywhere of weak limits of bi-Sobolev homeomorphisms

Abstract: This paper investigates the differentiability of weak limits of bi-Sobolev homeomorphisms. Given $p > n-1$, consider a sequence of homeomorphisms f_k with positive Jacobians $J_{f_k} > 0$ almost everywhere and $\sup_m (||f_k||_{W^{1,n-1}} + ||f_k^{-1}||_{W^{1,p}} < \infty$. We prove that if f and h are weak limits of f_k and f_k^{-1}, respectively, with positive Jacobians $J_f > 0$ and $J_h > 0$ a.e., then $h(f(x)) = x$ and $f(h(y)) = y$ both hold a.e. and f and h are differentiable almost everywhere.

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Rearrangement-invariant hulls of weighted Lebesgue spaces

Abstract: We characterize the rearrangement-invariant hull, with respect to a given measure μ, of weighted Lebesgue spaces. The solution leads us to first consider when this space is contained in the sum of $(L^1 + L^\infty)(\mathbb{R}, \mu)$ and the final condition is given in terms of embeddings for weighted Lorentz spaces.

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Optimal Gagliardo–Nirenberg interpolation inequality for rearrangement invariant spaces

Abstract: We prove optimality of the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_X \lesssim \|\nabla^2 u\|_Y^{1/2} \|u\|_Z^{1/2},$$

where Y, Z are rearrangement invariant Banach function spaces and $X = Y^{1/2} Z^{1/2}$ is the Calderón–Lozanovskii space. By optimality, we mean that for a certain pair of spaces on the right-hand side, one cannot reduce the space on the left-hand, remaining in the class of rearrangement invariant spaces. The optimality for the Lorentz and Orlicz spaces is given as a consequence, exceeding previous results. We also discuss pointwise inequalities, their importance and counterexample prohibiting an improvement.

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Characterization of functions with zero traces via the distance function and Lorentz spaces

Abstract: Consider a regular domain $\Omega \subset \mathbb{R}^N$ and let $d(x) = \operatorname{dist}(x, \partial\Omega)$. Denote $L_a^{1,\infty}(\Omega)$ the space of functions from $L^{1,\infty}(\Omega)$ having absolutely continuous quasinorms. This set is essentially smaller than $L^{1,\infty}(\Omega)$ but, at the same time, essentially larger than a union of all $L^{1,q}(\Omega)$, $q \in [1, \infty)$.

A classical result of late 1980's states that for $p \in (1, \infty)$ and $m \in \mathbb{N}$, u belongs to the Sobolev space $W_0^{m,p}(\Omega)$ if and only if $u/d^m \in L^p(\Omega)$ and $|\nabla^m u| \in L^p(\Omega)$. During the consequent decades, several authors have spent considerable effort in order to relax the characterizing condition. Recently, it was proved that $u \in W_0^{m,p}(\Omega)$ if and only if $u/d^m \in L^1(\Omega)$ and $|\nabla^m u| \in L^p(\Omega)$. In this paper we show that for $N \geq 1$ and $p \in (1, \infty)$ we have $u \in W_0^{1,p}(\Omega)$ if and only if $u/d \in L_a^{1,\infty}(\Omega)$ and $|\nabla u| \in L^p(\Omega)$. Moreover, we present a counterexample which demonstrates that after relaxing the condition $u/d \in L_a^{1,\infty}(\Omega)$ to $u/d \in L^{1,\infty}(\Omega)$ the equivalence no longer holds.

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An elementary proof of Acerbi Fusco minimizer existence theorem

Abstract: The weak lower semicontinuity of the functional

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx$$

is a classical topic that was studied thoroughly. It was shown that if the function f is continuous and convex in the last variable, the functional is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$. However, the known proofs use advanced instruments of real and functional analysis. Our aim here is to present proof that can be easily understood by students familiar only with the elementary measure theory.

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A sharp variant of the Marcinkiewicz theorem with multipliers in Sobolev spaces of Lorentz type

Abstract: I will exhibit an offprint of my recent joint paper with L. Grafakos and M. Mastyło, in which we establish a sharp variant of the classical Marcinkiewicz theorem on boundedness of Fourier multiplier operators. Our conditions on the symbol of the Fourier multiplier operator are formulated in terms of fractional Sobolev spaces; in particular, we make use of a certain fractional Lorentz-Sobolev space with a logarithmic weight.

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Typical behaviour of 1-Lipschitz images of n-rectifiable metric spaces

Abstract: Suppose X is a complete metric space and E is its $\mathcal{H}^n$-measurable subset. We say that E is n-rectifiable, if it can be covered, up to an $\mathcal{H}^n$-null set, with a countable number of Lipschitz images $g_i(F_i)$ of subsets F_i of $\mathbb{R}^n$. It was recently shown that there is a characterisation of rectifiability in the spirit of Besicovitch-Federer, via the (complete) metric space $(\mathrm{Lip}_1(X, \mathbb{R}^n), \|\cdot\|_\infty)$ of 1-Lipschitz functions $f\colon X \to \mathbb{R}^n$. If E is n-rectifiable, then a typical f in $\mathrm{Lip}_1(X, \mathbb{R}^n)$ satisfies $\mathcal{H}^n(f(E)) > 0$, while if E intersects every n-rectifiable set in an $\mathcal{H}^n$-null set (i.e. E is n-purely unrectifiable), a typical f satisfies $\mathcal{H}^n(f(E)) = 0$. A particular improvement of this result on the rectifiable side is of interest to us, namely, whether, assuming E is rectifiable, there is some $\Delta > 0$ such that a typical $f \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ (for some fixed $m > n$) satisfies $\mathcal{H}^n(f(E)) > \Delta$. The answer turns out to be complicated. If X is Euclidian then a typical f satisfies even $\mathcal{H}^n(f(E)) = \mathcal{H}^n(E)$ (i.e. $\Delta = \mathcal{H}^n(E)$), but no such $\Delta > 0$ exists for general metric spaces (already for $X = \mathbb{R}^n_\infty$). We discuss sufficient and necessary conditions on the metric space X such that a $\Delta > 0$ having the property described above exists.