Title:

Exponential growth in 2d: function spaces, optimal embeddings and critical PDE's

Abstract:

For domains $\Omega \subset \mathbb{R}^2$ the Sobolev space $H_0^1(\Omega)$ embeds into any L^p-space, and so one may look for a higher than polynomial growth for an optimal embedding. The Trudinger-Moser inequality states that there is an optimal embedding into the Orlicz-space $L_\varphi(\Omega)$, with maximal growth function $\varphi(s) = e^{s^2}$. We will discuss various forms of related inequalities.

In $\mathbb{R}^N$, $N \geq 3$, the best Sobolev embedding constants are known, and are never attained for bounded domains. To the contrary, in dimension 2 the best embedding constant is not known and, surprisingly, it was shown by Carleson-Chang (1986) that it is attained (if Ω is a disk). This implies that the associated Euler-Lagrange differential equation has a solution. We will present various existence results for related elliptic equations, which all present the difficulty of *loss of compactness* (see e.g. de Figueiredo-Miyagaki-R.,1995).

The fact that the supremum in the TM-inequality is attained lead Struwe (1988) to suspect, and prove, that the global maximum "survives" as a local maximum in the slightly supercritical case (in which the global supremum becomes infinite). This in turn lead to the search of mountain-pass solutions in the slightly super-critical case, which *blow up* as the super-critical exponent approaches the critical exponent from above (del Pino-Musso-R., 2010).

Beside the H^1-solutions (which are in fact regular), elliptic equations with exponential nonlinearities admit also *singular* solutions. They are *distributional* solutions which lie just barely outside of H^1 (and hence are not weak solutions). We discuss the construction of such solutions, and then show that they give rise to *non-uniqueness* for the associated heat equation (Fujishima-Ioku-R.-Terraneo, 2024).