An introduction to first-order calculus in non-smooth spaces

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In this course we will give an introduction to Sobolev space-theory for non-smooth metric measure spaces where the notion of weak derivatives is unavailable. We will discuss the notion of upper gradients inspired by Riemannian geometry, and discuss geometric properties of metric measure spaces of bounded geometry.

Lecture 1: p-moduli of families of curves, and ACL_p condition.
In this lecture, after a brief introduction to path integrals, we will discuss notions of p-modulus of families of curves. We will then discuss the Euclidean setting, and the property of *absolute continuity on p-modulus almost every line segment* parallel to the coordinate axes (ACL_p-property). We will also discuss the Beppo-Levi characterization of Sobolev functions.

Lecture 2: Upper gradients and their properties.
In this lecture, we will discuss the notions of upper gradients and p-weak upper gradients of measurable functions on a metric measure space X. We will discuss what the analogs of the classical differentiation rules are in this setting, and construct Newton-Sobolev spaces $N^{1,p}(X)$. We will discuss the Banach lattice property of $N^{1,p}(X)$, and consider some example non-smooth spaces as well.

Lecture 3: Poincaré inequalities.
In this lecture, we will talk about the main equivalence class property and Poincaré inequalities, and discuss what the geometric implications are of a metric measure space that supports a Poincaré inequality. In this lecture we will assume for the most part that the measure is doubling (I will explain what that means as well).

Lecture 4: Other candidate Sobolev spaces in non-smooth setting.
In this lecture, we will discuss alternate function-spaces such as Hajłasz-Sobolev spaces, Poincaré-Sobolev spaces, Korevaar-Schoen Sobolev spaces, and time permitting, we will also discuss the theory of Dirichlet forms briefly.